# Completely hereditarily atomic OMLs – Part II

John Harding, Andre Kornell

New Mexico State University
Dalhousie University
See math.nmsu.edu

jharding@nmsu.edu

SSAOS, Stara Lesna, September 2023

# Completely hereditarily atomic OMLs

Our results are purely order-theoretic, but motivation comes from quantum predicate logic. We would like infinite-height OMLs with some properties of $\mathcal{C}(\mathbb{C}^n)$.

- Complete, atomic, irreducible
- Algebraic (ALG)
- Covering property (CP)
- Every complete subalgebra is atomic (CHA)

View this talk as

- Exploring relationships among some properties of OMLs
- Providing a general construction of some interest
- Providing several non-trivial examples

# Definitions

For a lattice L and elements $k, x, y \in \mathrm{L}$ we say

| | |
|---|---|
| Compact | $k \leq \bigvee S \Rightarrow k \leq \bigvee S'$ for some finite $S' \subseteq S$ |
| ALG | Complete and each element is the join of compact ones |
| CHA | Complete and each complete subalgebra is atomic |
| Cover | $x < y$ and there is no $z$ with $x < z < y$ |
| CP | If $a$ is an atom, $x \in \mathrm{L}$, and $a \nleq x$ then $x \vee a$ covers $a$ |
| WA | Each interval contains a cover |
| MCWA | Each maximal chain is weakly atomic |

# Some examples

## Example $\mathcal{C}(\mathbb{C}^n)$

This OML is finite-height. Thus it is complete and atomic. Also, each element is compact, hence it is ALG. It is modular, and therefore has CP. It is obviously CHA and MCWA. It is further directly irreducible. It is wonderful, except it is of finite height.

## Example $\mathcal{C}(\mathrm{H})$ for H an infinite-dimensional Hilbert space

This OML is irreducible, complete, and atomic. It is not modular, but does have CP.

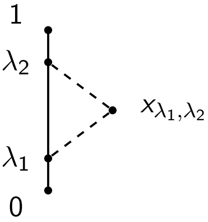
It is not ALG since no atom $\langle v \rangle$ is compact. Take an ONB $(u_n)_{\mathbb{N}}$ where $v$ is not a finite linear combination. Then $\bigvee \langle u_n \rangle = 1$ but $\langle v \rangle$ is not beneath a finite join.

It is not CHA since it has an atomless block. This is the beautiful curse of operators without pure point spectrum,.

## Some examples

Example It is instructive to see another instance of a complete atomic OML that has an atomless block.

Consider the lattice L below. It is built by starting with the real unit interval [0,1] and adding elements $x_{\lambda_1,\lambda_2}$ for each $\lambda_1 < \lambda_2$.




Then K(L) is atomic but has an atomless block. The same is true for its MacNeille completion, which is an OML since L is complete.

# Algebraic Boolean algebras

Proposition For a complete Boolean algebra these are equivalent

1. Atomic
2. Isomorphic to Pow(X)
3. Completely distributive $\bigvee_I \bigwedge_{J_i} a_{ij} = \bigwedge_\alpha \bigvee_I a_{i,\alpha(i)}$
4. ALG
5. CHA
6. MCWA

Example Pow $(\mathbb{N})$ is complete, atomic, modular, infinite height, has CP, is ALG, CHA and MCWA. Its drawback is that it is reducible.

Proposition For L a complete OML

$$\text{ALG} \Rightarrow \text{CHA} \Leftrightarrow \text{blocks are atomic} \Leftrightarrow \text{MCWA}$$

Proof. CHA $\Rightarrow$ blocks are atomic: Blocks are complete subalgebras.

blocks are atomic $\Rightarrow$ MCWA: Chains are contained in blocks.

MCWA $\Rightarrow$ CHA:

S $\leq$ L complete, u $\in$ S

D max chain in S containing u

C max chain in L containing D

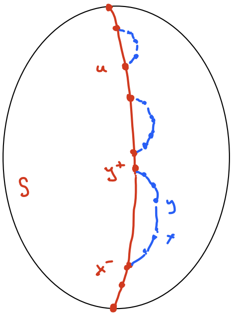
$x < y$ cover in [0,u] of C

$x^-$ largest in S under $x$

$y^+$ least in S above $y$

$x^- < y^+$ cover in S under u

$(y^+) \wedge (x^-)'$ atom of S under u




ALG $\Rightarrow$ blocks are atomic: A complete sublattice of ALG is ALG.

CHA $\not\Rightarrow$ ALG: Horizontal sum of 2 copies of Pow($\mathbb{N}$).

# Covering property

Proposition Each CHA, resp. ALG, OML with CP is a direct product of irreducible CHA, resp. ALG, OMLs with CP.

Proof. The center C(L) is a complete subalgebra, hence atomic.

Theorem For a complete irreducible OML,

$$\text{ALG} + \text{CP} \Leftrightarrow \text{modular} + \text{finite height}$$

Proof "$\Leftarrow$" Trivial. "$\Rightarrow$" If height $\leq 3$ then can't have both CP and a pentagon. If height $\geq 4$ then L $\simeq \mathcal{C}$(E) for a Hermitian space E. If E is finite-dimensional we are done. Else, E has a max orthogonal set $F = (f_\alpha)_\beta$ with $\beta$ an infinite ordinal. Then $\bigvee_\beta \langle f_\alpha \rangle = F^{\perp\perp} = 1$. Since L is algebraic, each $\langle v \rangle$ is under a finite join, so $F$ is a basis. Then $(f_\alpha - f_{\alpha+1})_\beta^{\perp\perp} = \text{E}$ so $\bigvee_\beta (f_\alpha - f_{\alpha+1}) = 1$. Hence $(f_\alpha - f_{\alpha+1})_\beta$ is also a basis. Impossible.

# Sharpness

For an irreducible OML of infinite height, ALG + CP is impossible.

We weaken our reqiuirements

ALG $\subseteq$ CHA

CP $\subseteq$ 2-CP

Definition L has 2-CP if $a$ atom, $x \in \mathrm{L} \Rightarrow [x, x \vee a]$ has height $\leq 2$.

Theorem

1. There is an irreducible infinite height OML with ALG + 2-CP
2. There is an irreducible infinite height OML with CHA + CP

# Extending Kalmbach's construction

We can't use K(L) directly to make ALG OMLs since it is almost never complete. We extend this construction.

Note The idea is that for the chain $C = \mathbb{N} \cup \{\infty\}$, the MacNeille completion of B(C) is Pow($\mathbb{N}$) and elements of Pow($\mathbb{N}$) are given by arbitrary unions of half-open intervals [m,n) of $\mathbb{N} \cup \{\infty\}$.

Definition An $\omega + 1$ lattice is a lattice where each maximal chain is isomorphic to $\mathbb{N} \cup \{\infty\}$.

Definition For L an $\omega + 1$ lattice, let $K^*(L)$ be all strictly increasing sequences in L that are either finite and of even length or infinite. Put an order and orthocomplement on $K^*(L)$ similar to K(L).

# Properties of $K^*(L)$

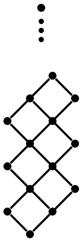
Theorem If L is an $\omega + 1$ lattice, then

1. $K^*(L)$ is a complete OL
2. $K^*(L)$ is the MacNeille completion of $K(L)$
3. $K^*(L)$ is an OML
4. Blocks of $K^*(L)$ are the $K^*(C)$ where C is a max chain of L
5. $K^*(L)$ is CHA

Proof (1) Effort, we won't do this. (2) $K(L)$ is join-dense in the complete OL $K^*(L)$. (3) An $\omega + 1$ lattice is complete. So the MacNeille completion of $K(L)$ is an OML. (4) If $x, y \in K^*(L)$ and have incomparable terms, then the commutator $\gamma(x, y) \neq 0$. (5) The blocks $K^*(C)$ are all isomorphic to $\mathrm{Pow}(\mathbb{N})$.

# An example of ALG + 2-CP

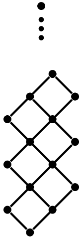
The following is the Reiger-Nishimura lattice.



Theorem For L the Reiger-Nishimura lattice, $K^*(L)$ is a complete, irreducible OML of infinite height that is ALG + 2-CP.

**Idea of the proof** For ALG, it is enough to show if a is an atom, S is a set of atoms, and a $\leq \bigvee$ S, then a $\leq \bigvee$ S' for some finite S' $\subseteq$ S. Key is that all but finitely many atoms are orthogonal to a. There is still effort here in closely examining things.

For 2-CP it is enough to show $[y, x \vee y]$ has height at most 2 when $x$ is an atom and $y$ is a join of atoms that do not commute with $x$. There aren't many atoms that don't commute with $x$.



## An example of CHA + CP

Recall from the tutorial ...

| | |
|---:|---|
| ordered abelian group | $\Gamma$ |
| field | $k$ |
| valuation | $\varphi : k \to \Gamma \cup \{\infty\}$ |
| ultra-metric | $d : k \times k \to \Gamma \cup \{\infty\}$ |

Here

$$\begin{aligned} \Gamma &= \bigoplus_{\mathbb{N}} \mathbb{Z} \text{ with reverse lexicographic order} \\ k &= \text{all generalized power series } x : \Gamma \to \mathbb{R} \\ \varphi(x) &= \text{least } \gamma \text{ in support}(x) \\ d(x,y) &= \varphi(x-y) \\ \gamma_n &= (0, \ldots, 1, \ldots) \end{aligned}$$

# An example of CHA + CP

Definition E is all sequences $f : \mathbb{N} \to k$ that are "square summable" in that $\sum f(n)^2 t^{\gamma_n}$ converges. Then $\langle \cdot | \cdot \rangle : \mathsf{E}^2 \to \Gamma \cup \{\infty\}$ is

$$\langle f \,|\, g \rangle = \sum f(n) g(n) t^{\gamma_n}$$

Recall that for any Hermitian space, $L_{\perp\perp}$ is a complete, irreducible, atomistic OL with CP.

Theorem (Keller) E is an orthomodular space, i.e. $\mathsf{L}_{\perp\perp}$ is an OML.

We show L is CHA

## An example of CHA + CP

We require a technique of Gross and Kunzie. Note that

$$\varphi\langle xf|xf\rangle = \varphi(x^2\langle f|f\rangle) = 2\varphi(x) + \varphi\langle f|f\rangle$$

So $\varphi\langle f|f\rangle/2\Gamma$ is constant on 1-dim subspaces.

Lemma $\varphi\langle f|f\rangle/2\Gamma = \gamma_n/2\Gamma$ for some $n \in \mathbb{N}$.

This gives T: E $\to \mathbb{N}$ that is constant on 1-dim subspaces.

Theorem (Gross, Kunzie)

1. If $f$ and $g$ are orthogonal, then $\mathrm{T}(f) \neq \mathrm{T}(g)$
2. $\{\mathrm{T}(f) : f \in \mathcal{M}\}$ is the same for each max orthogonal $\mathcal{M} \subseteq$ E.

# An example of CHA + CP

Theorem Each block B of $L_{\perp\perp}$ is atomic.

Proof (Idea) Define $\pi : B \to \mathrm{Pow}(\mathbb{N})$ as follows. For $S \in B$, take a maximal orthogonal set $\mathcal{M}$ of S and set

$$\pi(S) = \{T(f) : f \in \mathcal{M}\}$$

One can show that this is well-defined and a Boolean algebra isomorphism.

Question Is there an irreducible, complete, CHA OML with CP that has a block with uncountably many atoms? It is known that this will require different techniques.

# References

All can be found in the preprint available on ArXiv ...

J. Harding and A. Kornell, Completely hereditarily atomic OMLs.

Thank you